Correspondence Analysis for Visualizing Interplay of Pitch Class, Key, and Composer*

Hendrik Purwins, Thore Graepel,
Benjamin Blankertz, Klaus Obermayer

Abstract

We apply correspondence analysis for visualization of interdependence of pitch class & key and key & composer. A co-occurrence matrix of key & pitch class frequencies is extracted from score (Bach's WTC). Keys are represented as high-dimensional pitch class vectors. Correspondence analysis then projects keys on a planar "keyscape". Vice versa, on "pitchscapes" pitch classes can also be embedded in the key space. In both scenarios a homogenous circle of fifths emerges in the scapes. We employ biplots to embed keys and pitch classes in the keyscape to visualize their interdependence. After a change of co-ordinates the four-dimensional biplots can be interpreted as a configuration on a torus, closely resembling results from music theory and experiments in listener models.

In conjunction with spectral analysis, correspondence analysis constitutes a cognitive auditory model. Correspondence analysis of the co-occurrence table of intensities of keys and pitch classes lets the circle of fifths evolve in the pitchscape. This model works on digitized recorded music, does not require averaging or normalization of the data, and does not implicitly use circularity inherent in the model.

Statistics on key preference in composers yields a composer & key co-occurrence matrix. Then "stylescapes" visualize relations between musical styles of particular composers and schools. The Biplotting technique links stylistic characteristics to favored keys. Interdependence of composers and schools is meaningfully visualized according to their key preferences.

1 Introduction

The correspondence of musical harmony and mathematical beauty has fascinated mankind ever since the Greek idea of "the harmony of spheres". Of course, there exists a long tradition of analyzing music in mathematical terms. Vice versa, many composers have been inspired by mathematics. In addition, psychophysical experiments have been conducted, e.g., by Krumhansl

*in: *Perspectives in Mathematical Music Theory* (ed.: Emilio Luis-Puebla, Guerino Mazzola, and Thomas Noll), 2003, Epos-Verlag,Osnabrück

and Kessler (1982) to establish the relation between different major and minor keys in human auditory perception by systematic presentation of Shepard tones. The results of these experiments were visualized by a technique known as *multidimensional scaling* which allows to construct a two-dimensional map of keys with closely related keys close by. As a central result the *circle of fifths* (CoF) as one of the most basic tonal structures could be reproduced. In related work, a self-organizing feature map (Kohonen, 1982) of adaptive artificial neurons was applied to similar data, and showed, how the circle of fifths could be recovered by neural self-organization (Leman, 1995). In Leman and Carreras (1997) cadential chord progressions were embedded in a self-organizing feature map trained on Bach's "Well-Tempered Clavier" (WTC I). Based on that work, a cognitive model consisting of an averaged cq-profile [1] extraction (Purwins et al., 2000a) in combination with a self-organizing feature map revealed the circle of fifths after training on Alfred Cortot's recording of Chopin's Préludes.

In this paper we extend this general idea of embedding musical structure in two-dimensional space by considering the Euclidean embedding of musical entities whose relation is given in terms of a co-occurrence table. This general approach enables us not only to analyze the relation between keys and pitch-classes, but also of other musical entities including aspects of the style of composers. We can, for instance, exploit the fact that composers show strong preferences towards particular keys. This provides the basis for arranging the composers by correspondence analysis reflecting their stylistic relations.

According to Greenacre (1984), the interest in studying co-occurrence tables emerged independently in different fields such as algebra (Hirschfeld, 1935), psychometrics (Horst, 1935; Guttman, 1941), biometrics (Fisher, 1940), and linguistics (Benzécri, 1977). Correspondence analysis was discovered not only in distinct research areas but also in different schools, namely the pragmatic Anglo-American statistical schools as well as the geometric and algebraic French schools. Therefore, various techniques closely related to correspondence analysis have been discussed under various names, e.g., "reciprocal averaging", "optimal (or dual) scaling", "canonical correlation analysis of contingency tables", "simultaneous linear regressions".

We will first introduce the technique of correspondence analysis with a focus on the analysis of co-occurrences of keys and pitch-classes in Section 2. In Section 3 we will present the results of our correspondence analysis of inter-key relations in scores and recorded performances, that leads to the emergence of the circle of fifths and to a toroidal model of inter-key relations. We show how these results relate to a similar model from music theory (Chew, 2000) and to earlier experiments with a different cognitive model (Purwins et al., 2000a). In Section 4 we apply correspondence analysis to the problem of stylistic discrimination of composers based on their key preference. Finally, in Section 5 we point out some relations of our results to previous work and discuss potential application to other analysis tasks arising in music theory. Please note that we provide a more technical perspective on correspondence analysis in the Appendix, Section 6.

1 The abbreviation *cq* refers to *constant Q*, denoting a transformation with uniform resolution in the logarithmic frequency domain with a resulting constant ratio between frequency and band-width.

2 Analysis of Co-occurrence

Co-occurrence data frequently arise in various fields ranging from the co-occurrences of words in documents (information retrieval) to the co-occurrence of goods in shopping baskets (data mining). In the more general case, the co-occurring objects are chosen from two different sets. Correspondence analysis aims at embedding the objects in a lower-dimensional space such that the spatial relations in that space reflect the similarity of the objects as reflected by their co-occurrences.

Co-occurrence Table. Consider, as our running example, the co-occurrence table

$$\mathbf{H}^{\mathcal{K},\mathcal{P}} = (h_{ij}^{\mathcal{K},\mathcal{P}})_{\substack{1\le i\le 24\\ 1\le j\le 12}} \tag{1}$$

for keys ($\mathcal{K}$) and pitch classes ($\mathcal{P}$).

	c	...	...	b	$\mathbf{h}^{\mathcal{K}}$
C	$h_{\mathrm{C,c}}^{\mathcal{K},\mathcal{P}}$	...		$h_{\mathrm{C,b}}^{\mathcal{K},\mathcal{P}}$	$h_{\mathrm{C}}^{\mathcal{K}}$
...	...				...
B	...				$h_{\mathrm{B}}^{\mathcal{K}}$
Cm	...				$h_{\mathrm{Cm}}^{\mathcal{K}}$
...	...				...
Bm	$h_{\mathrm{Bm,c}}^{\mathcal{K},\mathcal{P}}$	...		$h_{\mathrm{Bm,b}}^{\mathcal{K},\mathcal{P}}$	$h_{\mathrm{Bm}}^{\mathcal{K}}$
$\mathbf{h}^{\mathcal{P}}$	$h_{\mathrm{c}}^{\mathcal{P}}$	...		$h_{\mathrm{b}}^{\mathcal{P}}$	n

Table $\mathbf{H}^{\mathcal{K},\mathcal{P}}$ reflects the relation between two sets $\mathcal{K}$ and $\mathcal{P}$ of events or objects (cf. Greenacre (1984)), in our case $\mathcal{K} = \{\mathrm{C}, \ldots, \mathrm{B}, \mathrm{Cm}, \ldots, \mathrm{Bm}\}$ being the set of different keys, and $\mathcal{P} = \{\mathrm{c}, \ldots, \mathrm{b}\}$ being the set of different pitch classes. Then an entry $h_{ij}^{\mathcal{K},\mathcal{P}}$ in the co-occurrence table would just be the number of occurrences of a particular pitch class $j \in \mathcal{P}$ in musical pieces of key $i \in \mathcal{K}$. The frequency $h_i^{\mathcal{K}}$ is the summation of occurrences of key i across all pitch classes. The frequency of pitch class j accumulated across all keys is denoted by $h_j^{\mathcal{P}}$. The sum of the occurrences of all pitch classes in all keys is denoted by n. From a co-occurrence table one can expect to gain information about both sets of objects, $\mathcal{K}$ and $\mathcal{P}$, and about the relation between objects in $\mathcal{K}$ and $\mathcal{P}$, i.e., between keys and pitch classes in the example above. The relative frequency of the entries is denoted by

$$f_{ij}^{\mathcal{K},\mathcal{P}} = \frac{1}{n} h_{ij}^{\mathcal{K},\mathcal{P}} . \tag{2}$$

It is the empirical joint distribution of $\mathcal{K} \times \mathcal{P}$. The relative frequency of column j is $f_j^{\mathcal{P}} = \frac{1}{n} h_j^{\mathcal{P}}$. It is the empirical marginal distribution of $f_{ij}^{\mathcal{K},\mathcal{P}}$. The diagonal matrix with $\mathbf{f}^{\mathcal{P}}$ on the diagonal is denoted $\mathbf{F}^{\mathcal{P},\mathcal{P}}$. The conditional relative frequency is denoted by

$$f_j^{\mathcal{P}|\mathcal{K}=i} = \frac{h_{ij}^{\mathcal{K},\mathcal{P}}}{h_i^{\mathcal{K}}} , \tag{3}$$

in matrix notation: $\mathbf{F}^{\mathcal{P}|\mathcal{K}} = (f_j^{\mathcal{P}|\mathcal{K}=\mathrm{i}})_{ij}$.

Instead of co-occurrence tables $\mathbf{H}^{\mathcal{K},\mathcal{P}}$ of frequencies of occurrences, in the sequel, we will also consider co-occurrence tables of overall symbolic durations (cf. Section 3.1) as well as co-occurrence tables of accumulated intensities (cf. Section 3.2).

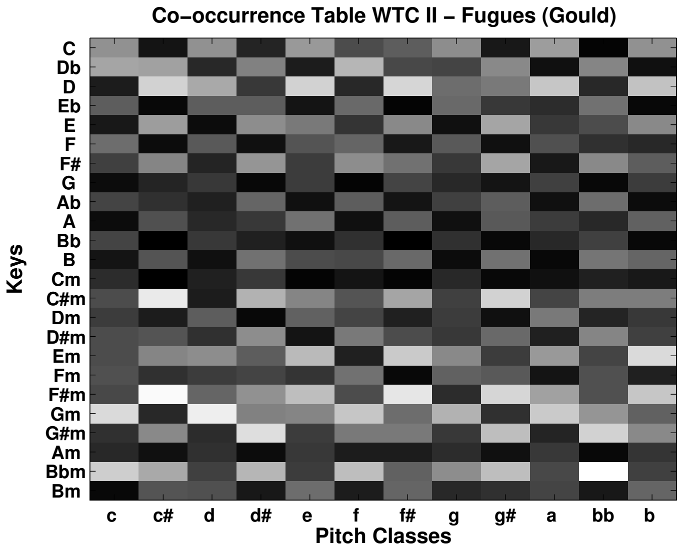

Figure 1: Co-occurrence table (cf. Table) of Bach's "Well-Tempered Clavier", Fugues of Book II recorded by Glenn Gould. The keys of the 24 Fugues are labeled on the vertical axis. For each fugue the intensities are accumulated for each pitch class, calculating cq-profiles (Purwins et al., 2000b). Light color indicates high intensity. Dark color indicates low intensity. This table is analyzed in Section 3.2.

2.1 Correspondence Analysis

Given a co-occurrence table $\mathbf{H}^{\mathcal{K},\mathcal{P}}$, for visualization purposes we aim at embedding the objects both in $\mathcal{K}$ and $\mathcal{P}$ in a two-dimensional space, such that aspects of their tonal relation are reflected by their spatial configuration. In particular, correspondence analysis can be thought of as a method that aims at finding a new co-ordinate system that optimally preserves the χ^2-distance between the frequency profiles of objects in $\mathcal{K}$ and $\mathcal{P}$, i.e., of rows and columns. For 12-dimensional pitch class frequency vectors $\mathbf{a}$ and $\mathbf{b}$ the squared χ^2-distance is defined by

$$\|\mathbf{a}-\mathbf{b}\|^2_{\mathcal{P}} := \langle \mathbf{a},\mathbf{a}\rangle_{\mathcal{P}} - 2\langle \mathbf{a},\mathbf{b}\rangle_{\mathcal{P}} + \langle \mathbf{b},\mathbf{b}\rangle_{\mathcal{P}} \tag{4}$$

with a generalized inner product defined by

$$\langle \mathbf{a},\mathbf{b}\rangle_{\mathcal{P}} := \mathbf{a}'(\mathbf{F}^{\mathcal{P},\mathcal{P}})^{-1}\mathbf{b}\,, \tag{5}$$

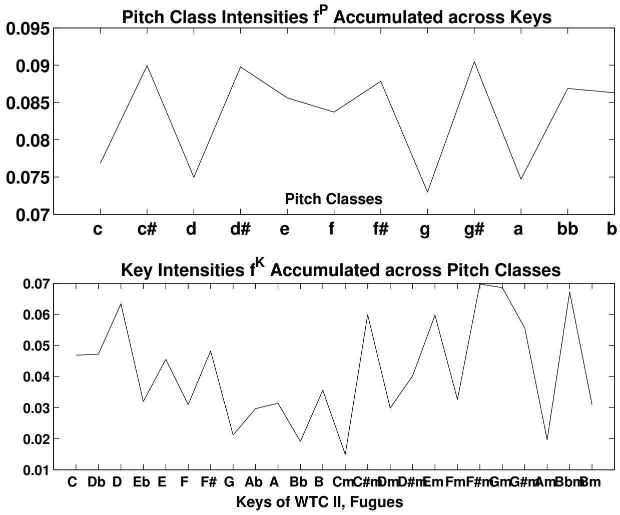

Figure 2: Relative frequency of pitch classes $\mathbf{f}^{\mathcal{P}}$ and keys $\mathbf{f}^{\mathcal{K}}$ of performed WTC II, Fugues, accumulated from the co-occurrence table (Figure 1). It is remarkable that the non-diatonic notes in C–Major are the most prominent notes, as if Bach wanted to oppose to the emphasis of C–Major in the mean tone tuning. *Upper:* $\mathbf{f}^{\mathcal{P}}$ is the normalized vector of pitch class intensities accumulated across all fugues in WTC II. *Lower:* $\mathbf{f}^{\mathcal{K}}$ is the normalized vector of accumulated intensity of each fugue.

where $\mathbf{a}'$ denotes the transpose of vector $\mathbf{a}$. The χ^2-distance is equal to the Euclidean distance in this example if all pitch classes appear equally often. The χ^2-distance weights the components by the overall frequency of occurrence of pitch classes, i.e., rare pitch classes have a lower weight than more frequent pitch classes. The χ^2-distance satisfies the natural requirement that pooling subsets of columns into a single column, respectively, does not distort the overall embedding because the new column carries the combined weights of its constituents. The same holds for rows.

We can explain correspondence analysis by a comparison to principal component analysis. In principal component analysis eigenvalue decomposition is used to rotate the co-ordinate system to a new one with the axes given by the eigenvectors. The eigenvalue associated with each eigenvector quantifies the prominence of the contribution of this particular co-ordinate for explaining the variance of the data. The eigenvector with highest eigenvalue indicates the most important axis in the data space: the axis with highest projected variance. Visualization in this framework amounts to projecting the high-dimensional data (the 12-dimensional pitch class frequency space or, respectively, the 24-dimensional key frequency space) onto a small number (typically 2 or 3) of eigenvectors with high eigenvalues. Hereby only insignificant dimensions of the data space are discarded, leading, effectively, to a plot of high-dimensional data in 2d or 3d space.

In principal component analysis by rotating the co-ordinate system, the

Euclidean distances between data points are preserved. Correspondence analysis is a generalization of principal component analysis: The χ^2 distance (a generalization of the Euclidean distance) between data points is preserved.

If the $m \times n-$ data matrix is not singular and not even symmetric, generalized singular value decomposition instead of eigenvalue decomposition yields two sets of *factors* $\mathbf{u}_1, \ldots, \mathbf{u}_d$ and $\mathbf{v}_1, \ldots, \mathbf{v}_d$ instead of one set of eigenvectors. So either for the m-dimensional column vectors of the data matrix the co-ordinate system can be rotated yielding a new co-ordinate system given by the column factors $\mathbf{u}_1, \ldots, \mathbf{u}_d$, or the n-dimensional row vectors of the data matrix are expressed in terms of co-ordinates in the new co-ordinate system of row factors $\mathbf{v}_1, \ldots, \mathbf{v}_d$. In principal component analysis each eigenvector is associated with an eigenvalue. In the same sense for each pair of column and row vectors u_k and v_k, an associated singular value δ_{kk} quantifies the amount of variance explained by these factors (see the appendix, Section 6 for technical details). In correspondence analysis the Euclidean distance in the two-dimensional projection of the data corresponds to the χ^2-distance in the 12-dimensional space of pitch frequencies, or respectively the 24-dimensional space of the key frequencies.

A *biplot* provides a simultaneous embedding of objects in $\mathcal{K}$ and $\mathcal{P}$. Both the co-ordinates of a $\mathcal{K}$-profile in the co-ordinate system of the u_k's and the co-ordinates of a $\mathcal{P}$-profile in the co-ordinate system of the v_k's are displayed in the same co-ordinate system. Such a biplot reveals the inter-set relationships.

3 Circle of Fifths in the Keyscape

3.1 Circle of Fifths from Score

Muse Data (CCARH, 2003) is a digital format that is aimed at fully and precisely representing the essential content of a composition as it is obvious from notation in a score. All fugues of Bach's WTC are encoded in Muse Data **kern format. Instead of analyzing the co-occurrence table of frequencies of keys and pitch classes we look at the *overall symbolic duration* of pieces in a particular key and the overall symbolic duration of pitch classes across all 24 Fugues for WTC I and for WTC II. Symbolic duration means that it is measured in multiples and fractions of quarter notes, rather than in seconds since Bach's scores do not provide information about the exact tempo of a performance.

The correspondence analysis of WTC (Figure 3) reveals a two–dimensional structure which allows for an adequate representation in a plot based on the first two factors corresponding to the two largest singular values (Figure 5). The 24 keys are embedded in the co-ordinate system of these factors such that the Euclidean distance between the projected keys in that plane optimally preserves the χ^2-distance between keys represented as 12-dimensional vectors of overall pitch class lengths. In the Fugues of WTC, the circle of fifths emerges clearly and homogeneously (upper Figure 3). Even though in the upper Figure 3 some inter-key distances are smaller (D♭–A♭) than others (D–A), due to different χ^2-distances between pitch class prominence profiles in these pieces. In addition the minor keys form a circle of fifths inside the circle of fifths of the major keys. This shows that the pitch prominence patterns for major keys are more distinct than for minor keys according to the metric employed.

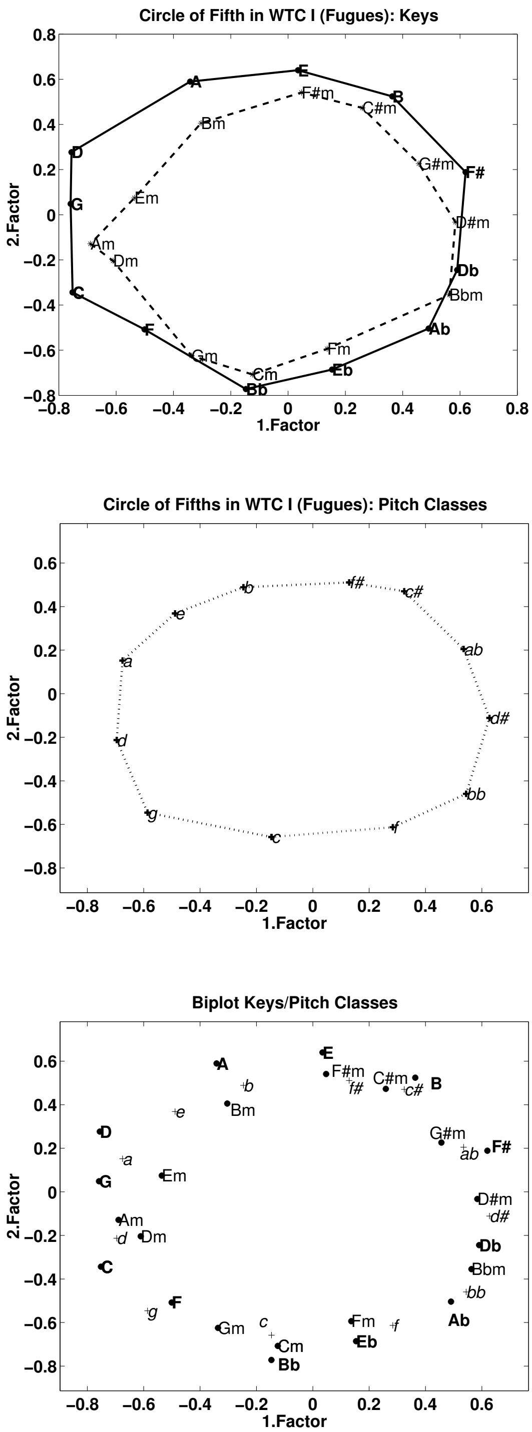

Figure 3: Symbolic durations of keys and pitch classes are derived from the score of the Fugues of Bach's WTC I and then projected onto the factors of correspondence analysis. *Upper*: The emerged circle of fifths is lined out for Major (solid) and minor (dashed) keys ("m" denoting minor). *Middle*: As above pitch classes appear in the order of the circle of fifths. *Lower:* The biplot of keys and pitch classes derived from putting both transparent upper graphs on top of each other. We observe the pitch classes being close to the minor keys that share the same fundamental.

In the co-occurrence table above pitch classes are represented as columns of accumulated symbolic durations in the different keys, that means accumulated symbolic durations in each fugue of WTC I, since in WTC I a one-to-one correspondence of Fugues and the 24 keys is given. The same factors with maximal singular values as in the upper Figure 3 are used to optimally project the 24-dimensional pitch class vectors upon a plane. We observe the pitch classes forming a circle of fifths as well (middle Figure 3). We can now consider the biplot (lower Figure 3) by putting the transparent plot of pitch classes (middle Figure 3) on top of the plot of keys (upper Figure 3). We have three circles of fifths, one each for the Major and minor keys and one for the pitch classes. We change the co-ordinates of the factor plane to polar co-ordinates in terms of a polar angle (on the circle of fifths) and the distance to the origin. Consider the angles of both the Major and minor keys relative to the angles of the fundamental pitch class (Figure 4). The plot shows two almost straight parallel lines, reflecting that pitch classes and keys proceed through the circle of fifths with almost constant offset. The graph for the minor keys is almost the identity, indicating that pitch classes and keys are "in phase": The pitch classes can be identified with the fundamentals of the minor keys. The explanation lies in the relatively high overall symbolic duration of the fundamental pitch class in the symbolic duration profile in minor compared to Major. Also the overall symbolic duration of the Fugues in minor is longer than the one in Major. We conclude that the minor keys induce the circle of fifths in the pitch classes.

In Figure 5 the singular values to the factors are shown. They indicate how much of the variance in the data is captured if correspondence analysis projects the data onto the factors with highest singular values. It is interesting that the explanatory values, e.g., the singular values, of the two most prominent factors in WTC I (Fugues) are almost equal, in contrast to the other singular values, whose explanatory value is much smaller.

Toroidal Model of Inter-Key Relations. Figure 3 displays the projection of keys and pitch classes onto the plane spanned by the two most prominent factors. How can we visualize the projection onto the first four factors? We represent points on each of the planes spanned by the first & second and third & fourth factor, respectively, in polar co-ordinates, i.e., by their polar angle and by their distance to the origin. We then plot their angle in the 1/2 plane against their angle in the 3/4 plane (upper Figure 6). Topologically, we can think of the resulting body as the surface of a torus, which can be parameterized by two angles. Upper Figure 6 can be viewed as a torus if we consider vertical and horizontal periodicity, i.e., we glue together the upper and lower side as well as the right and left side. The three circles of fifths then meander around the torus three times as indicated by the solid, dashed, and dotted lines. In addition to the relationship regarding fifths in upper Figure 6 we see that the projection on the 3. and 4. factor contains information about the inter relation between Major keys and their parallel and relative minor keys.

Consistency with a Geometric Model (Chew, 2000). It is fruitful to compare the toroidal interpretation (upper Figure 6) of the biplot of keys and pitch classes (lower Figure 3) with Chew (2000) (middle Figure 6). In Chew

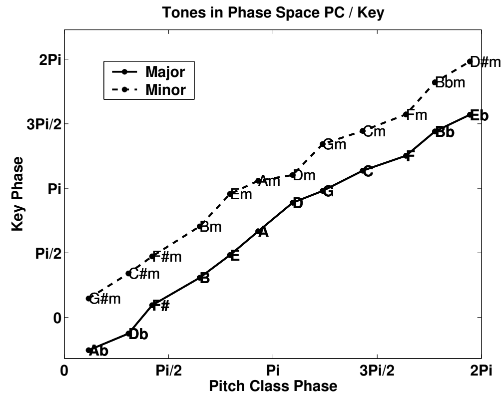

Figure 4: Phase of the circles of fifths described by Major and minor keys (upper Figure 3) relative to the phase of the circle of fifths in pitch classes (middle Figure 3). The graphs describe almost straight parallel lines. The angular co-ordinate of pitch classes and minor keys are "in phase". The angular co-ordinates of pitch classes and Major keys are offset by a constant value slightly less than $\pi/2$.

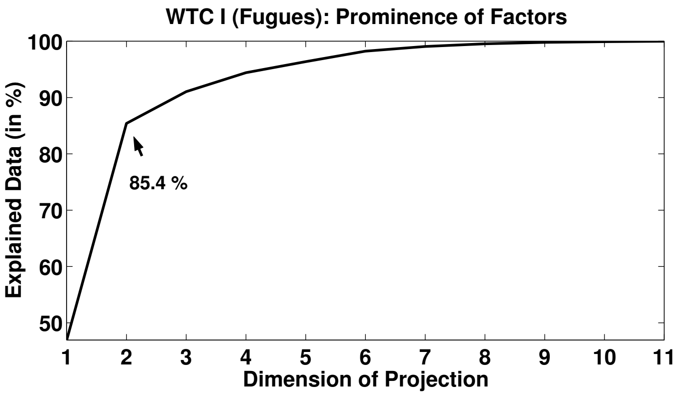

Figure 5: A low-dimensional representation is sufficient for representing the high-dimensional data: For WTC I (Fugues) in score representation the 2-dimensional projection of the overall symbolic duration of keys and pitch classes represents 85.4 % of the variance of the high dimensional vectors. Each of the two most prominent factors have approximately the same singular value.

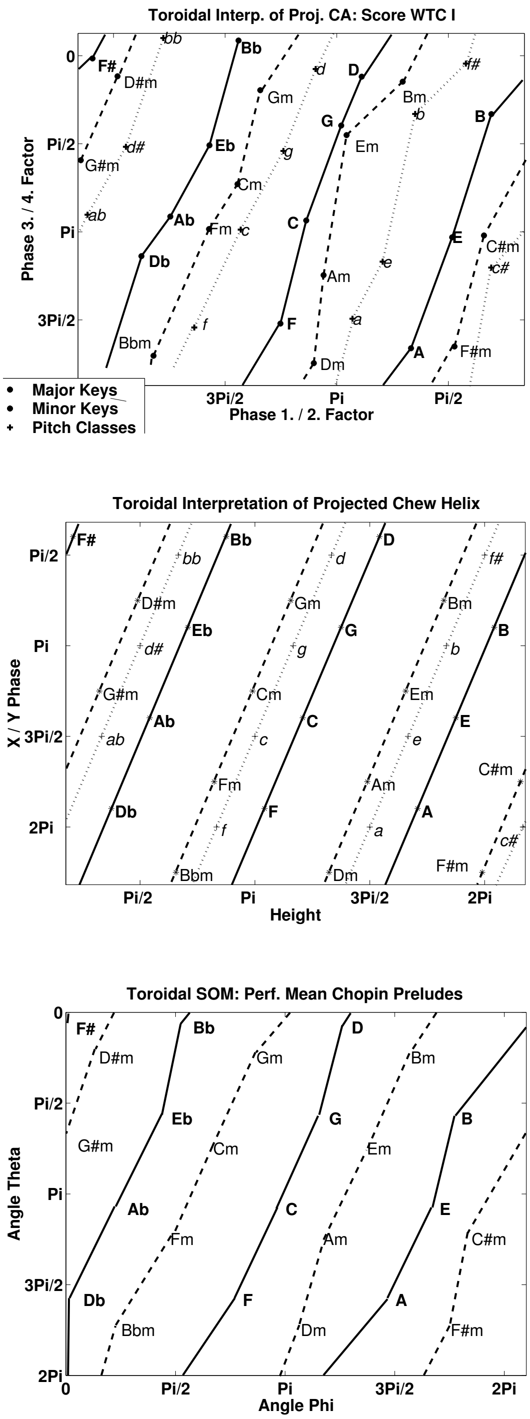

Figure 6: The toroidal interpretation of the projection on the first four factors (*upper*) is consistent with Chew (2000) (*middle*) and Purwins et al. (2000a) (*lower*). A torus can be described in terms of two angles, displayed on the horizontal and vertical axis. Glue the upper/lower and right/left side of the plots together to obtain a torus (cf. text).

(2000) heterogeneous musical quantities, namely, tones, chords, and keys are embedded in a three-dimensional space, thereby visualizing their inter relations (cf. Appendix 6.2 for technical details). The model is derived from the tonnetz (Euler, 1926; Lewin, 1987) and resembles Shepard's helix representation of pitches (Shepard, 1982). Tones are lined up on a helix along the circle of fifths, circular in the X/Y plane and elevating in the Z direction. For a triad composed of three tones, construct the triangle whose vertices are given by the tones constituting the triad. Then the triad is represented by the weighted center of gravity of the triangle. In the same way a key is represented as the center of gravity of the triangle whose vertices are the points given by the three main triads (tonic, dominant, subdominant) of the key. We reduce this model to pitch classes and keys, assuming that the intermediate level of chords is implicitly given in the music.

Chew (2000) gives a set of parameters derived by optimization techniques from musically meaningful constraints. We choose a different set of parameters to fit the model to the result of our correspondence analysis as displayed in upper Figure 6. (Cf. Appendix 6.2 for parameters.)

In order to facilitate comparison we choose a two-dimensional visualization of the three-dimensional model in Chew (2000). The projection of the model onto the X/Y plane is circular. Therefore we can parameterize it as angle and length. We plot the vertical dimension (the elevation of the helix) versus the phase angle of the X/Y plane (middle Figure 6). We interpret the phase angle of the X/Y plane as the first angle of a torus, and the vertical height in the helix as the second angle of a torus. We observe that upper and middle Figure 6 are very similar: The circles of fifths in Major and minor keys and in pitch classes curl around the torus three times. The only difference is that in the toroidal model derived from correspondence analysis Major keys and their relative minor keys are nearby, whereas in middle Figure 6 Major keys are closer to their parallel minor keys.

Consistency with a Cognitive Model (Purwins et al., 2000a). A very simple listener model comprises the following five stages:

1. Frequency analysis with uniform resolution on a logarithmic scale (constant Q transform of Brown (1991))
2. Compression into pitch class profiles by octave identification
3. Averaging of profiles across each piece
4. Generation of a reference set of profiles, one for each Major and minor key
5. Spatial arrangement of the reference set on a toroidal self-organizing feature map (Purwins et al., 2000b; Kohonen, 1982).

In this scheme, stage 1 can be considered a coarse model of auditory periphery. Stage 5 may be seen as a rough model of cortical feature maps (Obermayer et al., 1990). The constant Q transform is calculated from a digitized 1933/34 recording of Chopin's Préludes Op. 28 performed by Alfred Cortot. The average cq-profiles for each single prelude are used as a training set for a toroidal self-organizing feature map (Purwins et al., 2000b; Kohonen, 1982). Again the resulting configuration (lower Figure 6) shows the circle of fifths and closely resembles the other configurations in Figure 6.

3.2 Circle of Fifths in Performance

We choose a recording of the distinguished Glenn Gould playing the Preludes and Fugues of Bach's WTC, Book II. We calculate accumulated cq-profiles (Purwins et al., 2000b) from the set of the 24 Fugues of WTC II in all Major and minor keys (cf. Figure 1). Instead of containing frequencies of co-occurrences (cf. Table above) or symbolic durations (cf. Section 3.1 and Figure 3), the co-occurrence table now consists of the intensities of each pitch class accumulated for each of the 24 Fugues in all 24 Major and minor keys (Figure 1).

Pitch classes are represented by 24-dimensional key intensity vectors. In the same way as in Section 3.1, in correspondence analysis a singular value decomposition is performed yielding the factors as a new co-ordinate system. As in middle Figure 3, the pitch class vectors are projected onto a two-dimensional plane, spanned by the two most prominent factors. The circle of fifths evolves in pitch classes embedded in the keyspace in performance data as well. The two factors of performed WTC II (lower Figure 7) capture an even higher percentage (88.54 %) of the variance of the data, than those for the score data of WTC I (cf. Figure 5). Both factors are high and almost equal. Therefore the two–dimensional projection appears to be a very appropriate representation of pitch classes.

Comparisons have been made with other cycles of musical pieces like the Chopin's Préludes Op. 28 and Hindemith's "ludus tonalis": In these cycles one singular value alone is by far most prominent.

4 Stylescapes Based on Key Preference

Key Preference Statistics. In the following experiment the interplay of key preference and composer, rather than the interplay of key duration and pitch class duration is considered. For each key and for each composer the co-occurrence table now contains the number of pieces written in that particular key by that particular composer. Key preference statistics is counted in the following composers: J. S. Bach (JSB, only works for keyboard), L. v. Beethoven (LvB), J. Brahms (JB, non-vocal works), F. Chopin (FC), J. Haydn (JH), W. A. Mozart (WAM), A. Vivaldi (AV). If not stated otherwise, all works of the composer are considered, provided they contain the key name in the title of either the entire work or of single pieces, in case the work consists of a cycle of several pieces. For instance, a sonata in C Major is accounted for once, but WTC, Book I is accounted for 24 times. These key preference statistics (Figures 8 and 9) were generated from complete work lists of these composers found on the Internet.

Interplay between Tuning and Key Statistics. In the 16th and 17th century the mean tone tuning w.r.t. C was common. In this tuning the fifth g♯ – e♭ ("Wolfsquinte") sounds very rough, since this fifth is 35.7 cent (fourth of the syntonic comma) higher than the just fifth (Meister (1991) p. 81 cited in Groenewald (2003)). Bach supposedly mostly used Werckmeister (III) tuning, in which g♯ – e♭ still is the only fifth that is more than 2 cent larger than the just fifth, namely 5.9 cent (Dupont (1935) p. 81 cited in Groenewald (2003)). This is reflected in the low frequency of the usage of A♭ and g♯ that are only used in WTC. In the Kirnberger tuning (first version 1766 and third version

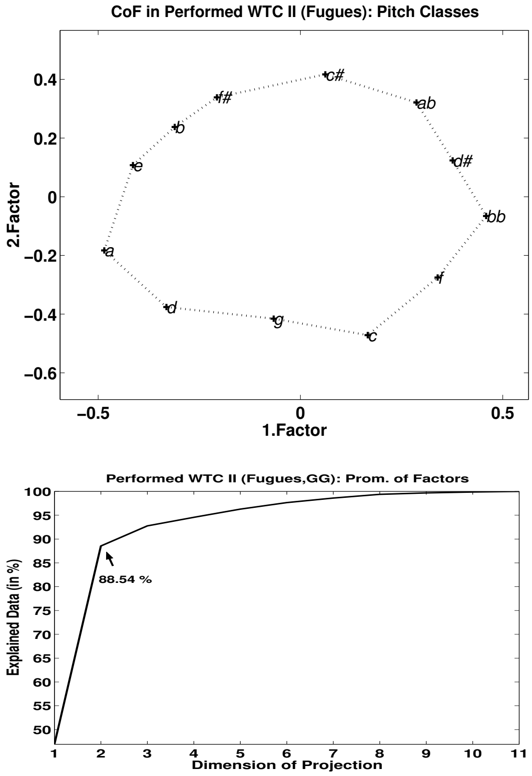

Figure 7: The circle of fifths (lined out) appears also in performed WTC (*upper*). The analyzed data are the overall intensities of pitch classes in the Fugues of Bach's WTC II in the recording of Glenn Gould shown in Figure 1. The same procedure as in Figure 3 (middle) is applied to project the 24-dimensional pitch class vectors onto a two-dimensional plane, spanned by the two most prominent factors. These two factors of performed WTC II capture an even higher percentage (88.54 %, *lower*) of the variance of the data than those for the score data of WTC I (cf. Figure 5).

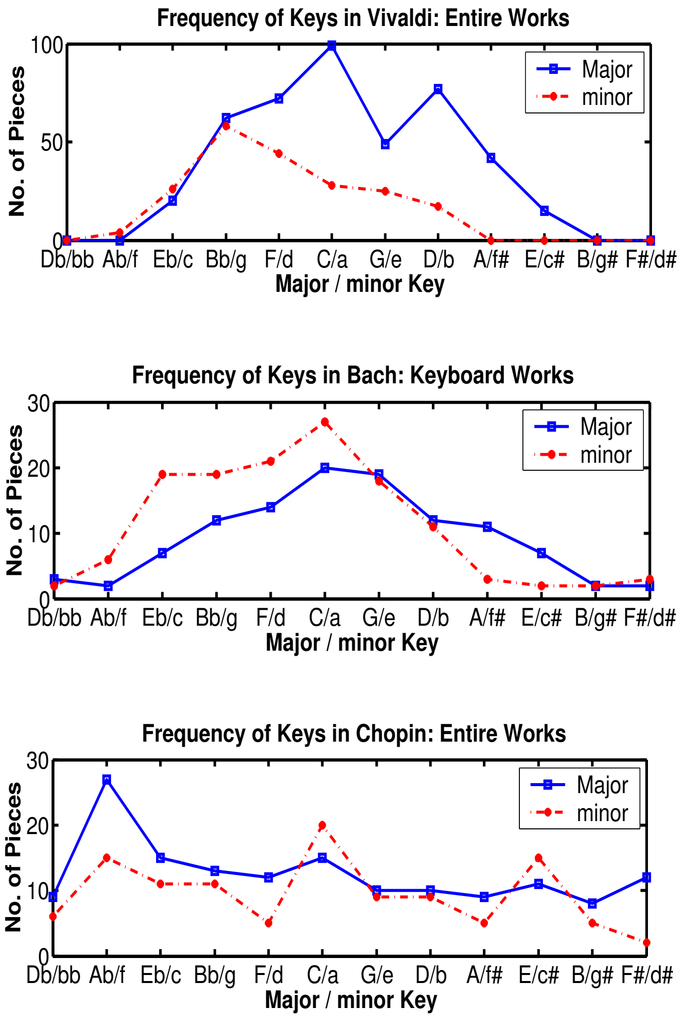

Figure 8: Key preference statistics in complete works of Vivaldi and Chopin, and the keyboard works of Bach. In this and the two subsequent Figures, small letters indicate minor keys. Vivaldi prefers C–Major and D–Major, Bach prefers a–minor, c–minor, and C–major. Some rare keys are only used in the WTC: F♯–Major,B–Major, A♭–Major, c♯–minor, g♯–minor, b♭–minor. 45.5% of the pieces are written in a Major key. The most frequent keys in Chopin are A♭–Major, a–minor, C–Major, E♭–Major, and c♯–minor. There are only two pieces in d♯–minor. 57% of the pieces are in Major.

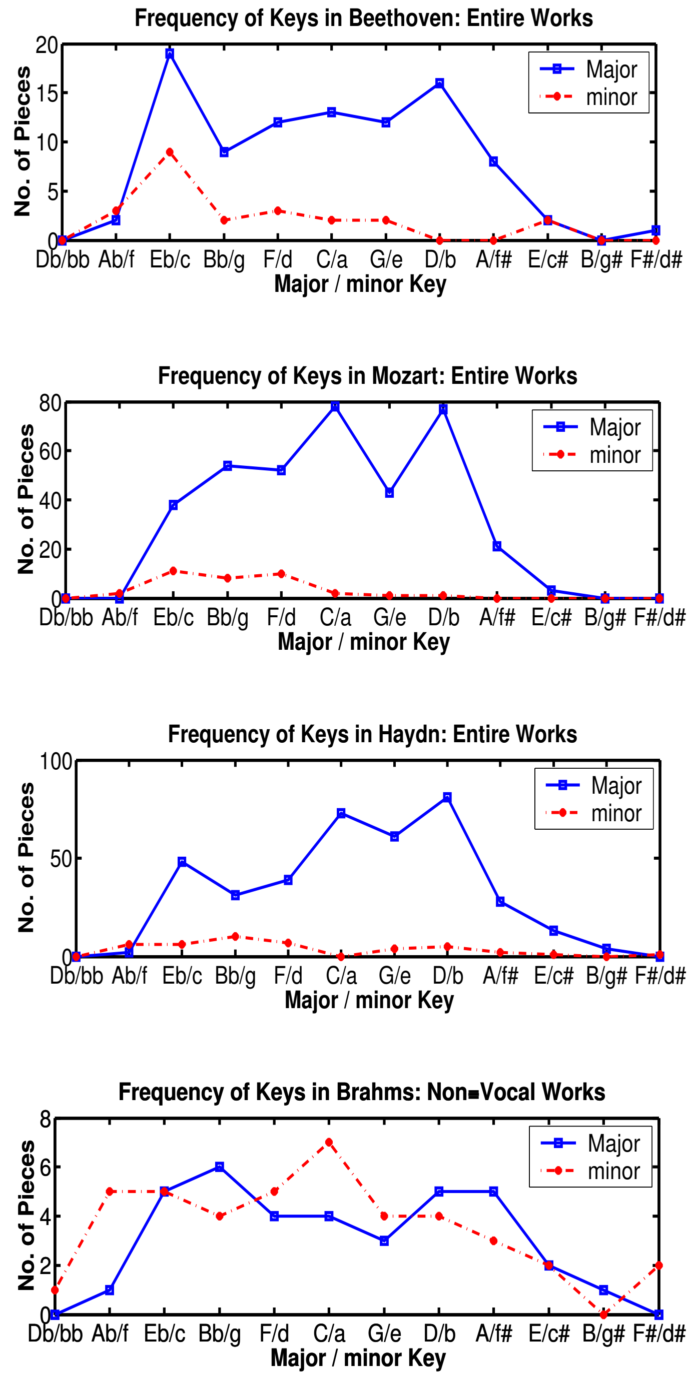

Figure 9: Beethoven prefers the "heroic" E♭–Major and D–Major. As in Vivaldi (cf. Figure 8) the key preference of Mozart and Haydn takes the "Cologne Dome" shape, favoring C–Major and D–Major. 90% of the keys are in Major.

1779), used by Chopin the fifth a♭ – e♭ is a just interval (Marpurg (1790) p. 21, Tessmer (1994) p. 199 cited in Groenewald (2003)). The prominence of A♭ in Chopin may be viewed as a result of the joy about the newly discovered terrain of keys that can be heard with comfort due to Kirnberger's new tuning.

Analysis. Correspondence analysis is performed on the key preferences of the seven composers. It noteworthy that correspondence analysis contributes to the varying amount of data available for the different composers. E. g. low number of available pieces for Brahms is not as important for correspondence analysis as the big number of pieces by Haydn. The projection of the composers into the 2-dimensional plane spanned by the two most prominent factors provides a stylescape: stylistically related composers are close to each other on this plane (cf. Figure 4). In the biplot composers/keys in Figure 4 composers are related to keys: Due to their shared "Cologne Dome" preference for C–Major and D–Major, Haydn and Mozart are very close and Vivaldi is not so far off. Beethoven is close to his favored E♭–Major and near to Haydn and Mozart. Brahms and Bach are positioned in their favored minor keys. Chopin maintains his outlier position due to the outlier position of his favored *A*♭–Major key. The explanatory value for the first (63.23 %) and the first two factors (88 %) in correspondence analysis is high. The most important factor is very dominant.

5 Discussion

In this paper we have shown how meaningful parameters in the complex structure of music can be visualized, by this revealing the inter relations of music looked upon in the perspective of a certain parameter. To demonstrate the high potential of this approach we have given examples in the domain of inter-key relations based on the perspective of looking at the frequency of pitch class usage and in the domain of stylistic categorization based on a view of the key preference of the different composers. The beauty of the method reveals since the approach is simple but non the less does require almost no assumptions, neither musical knowledge, nor special artificial data. The emergence of the circle of fifths has been observed in previous work on cognitive models. In Leman (1995) artificially generated cadential chord progressions constructed from Shepard tones are used as training data. Purwins et al. (2000a) used overall averaged digitized sound samples (Chopin's Préludes op. 28) for training. In contrast, in the present work we used accumulated vectors of each single digitized recording of the pieces in WTC as training vectors. In both Leman (1995) and Purwins et al. (2000a) the circular key structure is implicitly stipulated by the training of a *toroidal* self-organizing feature map. In the simulation discussed here the circularity emerges from the data alone, without an implicit assumption of periodicity in the model. In this sense, our analysis can be viewed as discovering a model of circular structure rather than merely fitting such a model.

The available data has not been exhaustively analyzed. Projections to different sub-planes could be explored and interpreted. The method can be used to model an experienced listener exposed to a new piece of music, and the listening experience in the context of a body of reference pieces. In correspon-

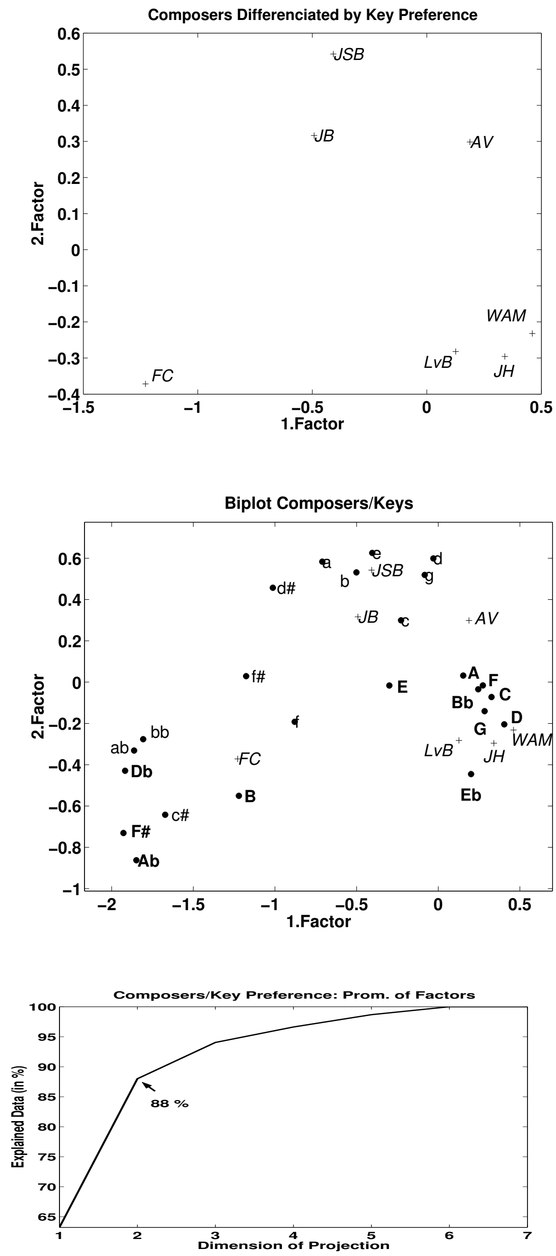

Figure 10: Based on the key preference profiles (cf. Figures 8 and 9) for different composers the stylescape (upper) and the biplot of composers/keys (middle) with associated singular values are shown. Capital letters indicate Major keys, small ones minor keys, italics indicate composers. *Upper:* Haydn (JH) and Mozart (WAM) are very close. Beethoven (LvB) is nearby. Brahms (JB) and Bach (JSB) can be considered a group. Chopin (FC) is an outlier. *Middle:* In the biplot composers/keys we observe that the Viennese classic (JH, WAM, LvB) gather in the region of D–Major and G–Major. Chopin (FC) maintains his outlier position due to the distant position of his favored A♭–Major key. *Lower:* The explanatory values for the first (63.23 %) and the two most prominent factors (88 %) in correspondence analysis are high.

dence analysis this would correspond to embedding the new pieces in a co-ordinate system obtained from analyzing the reference data. As an example, Bach's WTC has been used to generate a tonal co-ordinate system which then served to embed a number of other works including the Chopin's Préludes Op. 28, Alkan's Préludes, Scriabin's Préludes, Shostakovich's Préludes, and Hindemith's "ludus tonalis". In this way the method can be used to model how a listener who is familiar with Bach's WTC would perceive these keys and pitches in these more recent works. In addition, concepts of inter-key relations underlying Hindemith and Scriabin may be discovered.

We would like to emphasize that the use of correspondence analysis is by no means limited to tonality analysis. The method is a universal and practical tool for discovering and analyzing correspondences between various musical parameters that are adequately represented by co-occurrences of certain musical events or objects. Examples include pitch classes, keys, instrumentation, rhythm, composers, and styles. Three-dimensional co-occurrence arrays, for instance of pitch classes, keys, and metric positions can be analyzed. In particular, it seems promising to extend our analysis to temporal transitions in the space of musical parameters.

Acknowledgments

We would like to thank Ulrich Kockelkorn for advice in correspondence analysis and Hans-Peter Reutter for advice in music theory.

6 Appendix

6.1 Technical Details of Correspondence Analysis

In this appendix we provide some more technical details relating singular value decomposition and correspondence analysis. The following theorem is crucial for the reduced rank approximation of the co-occurrence matrix:

Theorem 1 (Generalized Singular Value Decomposition) *Let* $\mathbf{A}$ *be a positive definite* $m \times m$ *matrix and* $\mathbf{B}$ *a positive definite* $n \times n$ *matrix. For any real-valued* $m \times n$ *matrix* $\mathbf{F}$ *of rank* d *there exist an* $m \times d$ *matrix* $\mathbf{U} = (\mathbf{u_1}, \dots, \mathbf{u}_d)$, *a* $d \times n$ *matrix* $\mathbf{V} = (\mathbf{v_1}, \dots, \mathbf{v}_d)'$ *with* $\mathbf{U}'\mathbf{A}\mathbf{U} = \mathbf{I}_m$ *and* $\mathbf{V}'\mathbf{B}\mathbf{V} = \mathbf{I}_n$, *and a diagonal* $d \times d$ *matrix* $\boldsymbol{\Delta} = (\delta_{ij})$ *so that:*

$$\mathbf{F} = \mathbf{U}\boldsymbol{\Delta}\mathbf{V}' = \sum_{k=1}^{d} \delta_{kk}\mathbf{u}_k\mathbf{v}'_k. \tag{6}$$

Cf. Greenacre (1984) for a proof. For symmetric $\mathbf{F}$ the theorem yields the familiar special case of eigendecomposition.

The columns $\mathbf{u}_k$ of $\mathbf{U}$ can be viewed as the (column) factors, with singular values δ_{kk}. The magnitude of $\mathbf{F}$ in each of the d dimensions in the co-ordinate system spanned by the factors $\mathbf{u}_k$ is then given by δ_{kk}.

For $1 \leq d^* \leq d$ let $u_1, \ldots, u_{d^*}$ be the factors with the largest corresponding singular values $\delta_{11}, \ldots, \delta_{d^*d^*}$. Then

$$\mathbf{F}^* = \sum_{k=1}^{d^*} \delta_{kk} \mathbf{u}_k \mathbf{v}'_k \tag{7}$$

is the *rank d^* least squares approximation* of $\mathbf{F}$ (again cf. Greenacre (1984) for a proof). Of course, since the method is fully symmetric with respect to $\mathbf{U}$ and $\mathbf{V}$ we can also view the column vectors $\mathbf{v}_k$ of $\mathbf{V}$ as the (row) factors.

For the matrix of relative frequencies $\mathbf{F}^{\mathcal{P},\mathcal{K}} = (f_{ij}^{\mathcal{P},\mathcal{K}})$ and positive definite diagonal matrices $(\mathbf{F}^{\mathcal{P},\mathcal{P}})^{-1}$ and $(\mathbf{F}^{\mathcal{K},\mathcal{K}})^{-1}$ with the inverted relative frequencies of row and column objects, respectively, on their diagonal, Theorem 1 yields:

$$\mathbf{F}^{\mathcal{P},\mathcal{K}} = \mathbf{U}\boldsymbol{\Delta}\mathbf{V}', \tag{8}$$

with

$$\mathbf{U}'(\mathbf{F}^{\mathcal{P},\mathcal{P}})^{-1}\mathbf{U} = \mathbf{I}_m \quad \text{and} \quad \mathbf{V}'(\mathbf{F}^{\mathcal{K},\mathcal{K}})^{-1}\mathbf{V} = \mathbf{I}_n \tag{9}$$

The projection of the pitch class profile $\mathbf{f}^{\mathcal{P}|\mathcal{K}=\mathrm{i}}$ $(1 \leq i \leq 24)$ on a row factor $\mathbf{v}_k$ $(1 \leq k \leq d)$ yields the following coordinate (Kockelkorn, 2000):

$$s_{ki} = \langle \mathbf{v}_k, \mathbf{f}^{\mathcal{P}|\mathcal{K}=\mathrm{i}} \rangle_{\mathcal{P}}. \tag{10}$$

Vice versa for the projection of the key profile $\mathbf{f}^{\mathcal{K}|\mathcal{P}=\mathrm{j}}$ $(1 \leq j \leq 12)$ on a column factor $\mathbf{u}_k$ $(1 \leq k \leq d)$ the new coordinate is:

$$z_{kj} = \langle \mathbf{u}_k, \mathbf{f}^{\mathcal{K}|\mathcal{P}=\mathrm{j}} \rangle_{\mathcal{K}}. \tag{11}$$

6.2 Details of Chew's Model with Choice of Parameters

We use a simplified instance of Chew's more general model (Chew, 2000). It proposes a spatial arrangement such that tones, triads and keys are represented as vectors in three-dimensional space. For $j \in \mathbb{N}$ tones are denoted by $\mathbf{t}(j) \in \mathbb{R}^3$, proceeding in steps of one fifth interval from index j to index $j+1$. We denote major and minor triads by $\mathbf{c}_{\mathrm{M}}(j), \mathbf{c}_{\mathrm{m}}(j) \in \mathbb{R}^3$, and Major and minor keys by $\mathbf{k}_{\mathrm{M}}(j), \mathbf{k}_{\mathrm{m}}(j) \in \mathbb{R}^3$, respectively.

The tones are arranged in a helix turning by $\pi/2$ and rising by a factor of h per fifth:

$$\mathbf{t}(j) = \left(\sin\left(\frac{j\pi}{2}\right), \cos\left(\frac{j\pi}{2}\right), jh\right) \tag{12}$$

Both Major and minor triads are represented as the weighted mean of their constituent tones:

$$\begin{aligned} \mathbf{c}_{\mathrm{M}}(j) &= m_1\mathbf{t}(j) + m_2\mathbf{t}(j+1) + m_3\mathbf{t}(j+4) && (13)\\ \mathbf{c}_{\mathrm{m}}(j) &= m_1\mathbf{t}(j) + m_2\mathbf{t}(j+1) + m_3\mathbf{t}(j-3) && (14) \end{aligned}$$

The keys are represented as weighted combinations of tonic, dominant, and subdominant, with the minor keys additionally incorporating some of the Major chords:

$$\mathbf{k}_{\mathrm{M}}(j) = m_1\mathbf{c}_{\mathrm{M}}(j) + m_2\mathbf{c}_{\mathrm{M}}(j+1) + m_3\mathbf{c}_{\mathrm{M}}(j-1) \tag{15}$$

$$\mathbf{k}_{\mathrm{m}}(j) = m_1\mathbf{c}_{\mathrm{m}}(j) + m_2\left(\frac{3}{4}\mathbf{c}_{\mathrm{M}}(j+1) + \frac{1}{4}\mathbf{c}_{\mathrm{m}}(j+1)\right) \tag{16}$$

$$+ m_3\left(\frac{3}{4}\mathbf{c}_{\mathrm{m}}(j-1) + \frac{1}{4}\mathbf{c}_{\mathrm{M}}(j-1)\right). \tag{17}$$

We choose the parameters so as to obtain a good fit with the data from Bach's WTC I (Fugues), resulting in the following parameter settings:

$$\mathbf{m} = (m_1, m_2, m_3) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}) \text{ and } h = \frac{\pi}{6}. \tag{18}$$

The general model of Chew (2000) allows the weights to be independent from each other for $\mathbf{c}_{\mathrm{M}}, \mathbf{c}_{\mathrm{m}}, \mathbf{k}_{\mathrm{M}}, \mathbf{k}_{\mathrm{m}}$. In this application, weights $\mathbf{m}$ are set equal, across Major and minor chords and keys (cf. Chew (2001)). In our instance of the model, we have only $\mathbf{m}$ and h as free parameters. But we use values different from Chew (2001).

References

BENZÉCRI, J.-P. (1977). *Histoire et préhistoire de l'analyse des données*. *Cahiers de l'Analyse des Données*, 2:9–40.

BROWN, J. (1991). *Calculation of a constant Q spectral transform*. *J. Acoust. Soc. Am.*, 89(1):425–434.

CCARH (2003). *Muse Data*. Center for Computer Assisted Research in the Humanities. http://www.musedata.org.

CHEW, ELAINE (2000). *Towards a Mathematical Model for Tonality*. Ph.D. thesis, MIT Sloan School of Management.

CHEW, ELAINE (2001). *Modeling Tonality: Application to Music Cognition*. In *Proceedings of the 23rd Annual Meeting of the Cognitive Science Society*.

DUPONT, WILHELM (1935). *Geschichte der musikalischen Temperatur*. Bärenreiter-Verlag, Kassel.

EULER, L. (1926). *Opera Omnia*, vol. 1 of 3, chap. Tentamen novae theoriae musicae. Stuttgart.

FISHER, R. A. (1940). *The Precision of Discriminant Functions*. *Ann. Eugen.*, 10:422–429.

GREENACRE, M. J. (1984). *Theory and Applications of Correspondence Analysis*. Academic Press, London.

GROENEWALD, J. (2003). *128 musikalische Temperaturen im mikrotonalen Vergleich*. http://www.groenewald-berlin.de/.

GUTTMAN, L. (1941). *The Quantification of a Class of Attributes: A Theory and Method of Scale Construction*. In HORST, P. (ed.), *The Prediction of Personal Adjustment*. Social Science Research Council, New York.

HIRSCHFELD, H. O. (1935). *A Connection between Correlation and Contingency*. *Cambridge Philosophical Soc. Proc. (Math. Proc.)*, 31:520–524.

HORST, P. (1935). *Measuring Complex Attitudes*. *J. Social Psychol.*, 6:369–374.

KOCKELKORN, ULRICH (2000). *Multivariate Datenanalyse*. Lecture Notes.

KOHONEN, T. (1982). *Self-Organized Formation of Topologically Correct Feature Maps*. *Biol. Cybern.*, 43:59–69.

KRUMHANSL, C. L. and KESSLER, E. J. (1982). *Tracing the Dynamic Changes in Perceived Tonal Organization in a Spatial Representation of Musical Keys*. *Psychological Review*, 89:334–68.

LEMAN, M. (1995). *Music and Schema Theory*, vol. 31 of *Springer Series in Information Sciences*. Springer, Berlin, New York, Tokyo.

LEMAN, M. and CARRERAS, F. (1997). *Schema and Gestalt: Testing the Hypothesis of Psychoneural Isomorphism by Computer Simulation*. In LEMAN, M. (ed.), *Music, Gestalt, and Computing*, no. 1317 in Lecture Notes in Artificial Intelligence, pp. 144–168. Springer, Berlin.

LEWIN, DAVID (1987). *Generalized Musical Intervals and Transformations*. Yale University Press.

MARPURG, FRIEDRICH W. (1790). *Allerley Arten von Temperaturen*. Gottlieb August Lange, Berlin.

MEISTER, WOLFGANG THEODOR (1991). *Die Orgelstimmung in Süddeutschland vom 14. bis zum Ende des 18.Jahrhunderts*. Orgelbau-Fachverlag Rensch, Lauffen am Neckar.

OBERMAYER, K., RITTER, H., and SCHULTEN, K. (1990). *A Principle for the Formation of the Spatial Structure of Cortical Feature Maps*. *Proc. Natl. Acad. Sci. USA*, 87:8345–8349.

PURWINS, H., BLANKERTZ, B., and OBERMAYER, K. (2000a). *Computing Auditory Perception*. *Organised Sound*, 5(3):159–171. URL `ftp://ftp.cs.tu-berlin.de/pub/local/ni/papers/pur00b_OrgSound_CompAudPerc.ps.gz`.

PURWINS, H., BLANKERTZ, B., and OBERMAYER, K. (2000b). *A New Method for Tracking Modulations in Tonal Music in Audio Data Format*. In AMARI, S.-I., GILES, C.L., GORI, M., and PIURI, V. (eds.), *International Joint Conference on Neural Networks*, vol. 6, pp. 270–275. IJCNN 2000, IEEE Computer Society.

SHEPARD, R. N. (1982). *Geometrical Approximations to the Structure of Musical Pitch*. *Psychological Review*, 89:305–333.

TESSMER, MANFRED (1994). *Wie war Bachs Wohltemperirtes Clavier gestimmt?*, vol. 25 of *Acta Organologica*.